Rutgers University: Algebra Written Qualifying Exam
August 2014: Problem 5 Solution

Exercise. Let R be a commutative ring with 1. Let I and J be ideals in R such that for every $x \in R$ there is $y \in I$ such that $x \equiv y (\mod J)$. Show that for every $x \in R$ there is $z \in I$ such that $x \equiv z (\mod J^2)$. (Here J^2 is the ideal generated by all products rs, $r \in J$, $s \in J$.)

Solution.

Since $1 \in R$, $\exists y \in I$ such that $1 \equiv y \mod J$.

$\implies (1-y) \in J$

Then for any $x \in R, \exists v \in I$ s.t. $x \equiv v \mod J$.

$\implies (x-v) \in J$

So $\exists s, t \in J$ such that

$$\begin{aligned} & 1-y=x & \text{and} \quad & x-v=t \\ \implies & 1=y+s & & x=v+t \\ \implies & x=1\cdot x & & \\ & =(y+s)(v+t) & & \\ & =yv+sv+yt+st & & \\ & =z+st & \text{where} \quad & z=yv+sv+yt \end{aligned}$$

<u>**Note:**</u> $z \in I$ since $y, v \in I$

$\implies$ $yv, sv, yt \in I$ since $ri = ir \in I$ for any $i \in I$ and $r \in R$

$\implies$ $yv+sv+yt \in I$ since ideals are closed under addition

Also $s \in J$ and $t \in J$ $\implies$ $st \in J^2$

$\implies$ $x = z + st \equiv z \mod J^2$

Thus, since x was arbitrary,

$$\forall x \in R, \exists z \in I \text{such that } x \equiv z \mod J^2$$